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RANKING AND SELECTION IN DESIGNED EXPERIMENTS:
COMPLETE FACTORIAL EXPERIMENTS*

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Over the years, while it has been recognized that developments for other designs are of great interest, almost all ranking and selection papers have dealt explicitly only with the completely randomized design. We review what is known in other design settings, then proceed to new results on the complete factorial experimental design. We show that existing procedures are vitiated by interaction in such a design, but give a new procedure which is fully valid even in the presence of unbounded interaction.

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ABSTRACT

As early as the first paper of Bechhofer on ranking and selection in 1954 it was recognized that, with a ranking and selection goal as well as with other goals such as estimation or hypothesis testing, it might be desirable to carry out one's experiment in some design other than the completely randomized design. Nevertheless, over the years almost all of the papers in the area have developed their methods and theory explicitly only for the completely randomized design. In this paper we review what is known about ranking and selection in design settings other than the completely randomized design, and then proceed to new results on the complete factorial experiment setting.

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1. INTRODUCTION

In the first paper on ranking and selection, Bechhofer (1954) recognized the importance of allowing for experiments carried out in some design other than the completely randomized design. For full references see Dudewicz and Koo (1981). In this paper we: briefly survey the state of knowledge of ranking and selection in designed experiments (Section 2); discuss factorial experiments without interaction in some detail (Section 3); and give new results for factorial experiments where interaction may be present (Section 4).

2. RANKING AND SELECTION IN DESIGNED EXPERIMENTS

In the basic formulation of the ranking and selection problem, there are K populations (sources of observations) $\pi_1, \ldots, \pi_K$ with respective unknown means $\mu_1, \ldots, \mu_K$ for their observations, normally distributed with common known variance $\sigma^2 > 0$, and the goal is to select a population whose mean is $\mu_{[K]}$, where $\mu_{[1]} \leq \mu_{[2]} \leq \cdots \leq \mu_{[K]}$ are the ordered $\mu_1, \ldots, \mu_K$. Achieving this goal is called making a Correct Selection (abbreviated as CS). Bechhofer (1954) gave a single-stage procedure which guaranteed that $P(CS) \geq P^*$ whenever $\mu_{[K]} - \mu_{[K-1]} \geq \delta^* (1/K < P^* < 1, 0 < \delta^*)$. There have been many papers written in the area, with extensions in many directions; see Dudewicz (1980).

Bechhofer (1954) dealt mainly with the completely randomized design, but also considered the 2-factor factorial experiment without interaction, giving the Factorial Procedure SP1 of Section 3 below. He briefly noted how to similarly solve the r-factor factorial experiment (again without interaction) with SP1. Also see pp. 77-82 of Gibbons, Olkin, and Sobel (1977) for examples, restricted to the no-interaction case.

Bawa (1972) compared Bechhofer's SP1 to the traditional One-at-a-time Procedure SP2 in a no-interaction setting, and found SP1 is superior in asymptotic efficiency.

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A procedure for the factorial experiment with interaction (Interaction Procedure SP3 in Section 3 below) was given in Dudewicz (1977), and independently simultaneously by Bechhofer (1977).

For the problem of selecting the best regression, Dudewicz (1976, Section 14.4) noted the solution (also see Chapter 9 of Gibbons, Olkin, and Sobel (1977), with an erroneous claim of originality on p. 241). The problem of selecting the largest interaction was addressed by Bechhofer, Santner, and Turnbull (1977). Rinott and Santner (1977) applied inequalities to design an experiment to select the best treatment in an analysis of covariance model.

3. RANKING AND SELECTION IN FACTORIAL EXPERIMENTS: NO INTERACTION

Suppose an r-factor ($r \geq 2$) factorial experiment with k_j ($k_j \geq 2$) levels of factor j (j = 1, 2, ..., r). Assume observations are normally distributed with known common variance $\sigma^2 > 0$ and that for an observation taken at {level i_1 of factor 1, level i_2 of factor 2, ..., level i_r of factor r}

$$E(X_{i_1 i_2 \cdots i_r}) = \mu + \sum_{j=1}^{r} \alpha_{i_j}^{(j)}$$

(i_j = 1, 2, ..., k_j; j = 1, 2, ..., r) where μ and $\alpha_{i_j}^{(j)}$ are unknown parameters with

$$\sum_{i_j=1}^{k_j} \alpha_{i_j}^{(j)} = 0 \qquad (j = 1, 2, \ldots, r).$$

For each fixed j (j = 1, 2, ..., r), let

$$\alpha_{[1]}^{(j)} \leq \alpha_{[2]}^{(j)} \leq \cdots \leq \alpha_{[k_j]}^{(j)}$$

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denote the $a_1^{(j)}, \ldots, a_{k_j}^{(j)}$ in numerical order. Each factor-level combination $(i_1, i_2, \ldots, i_r)$ is called a population, and a "best" population (among all $K = k_1 k_2 \ldots k_r$ populations) is any one with maximum $E(X_{i_1 i_2 \ldots i_r})$, i.e. any one with mean

$$\mu_{[K]} = \mu + a_{[k_1]}^{(1)} + a_{[k_2]}^{(2)} + \ldots + a_{[k_r]}^{(r)} .$$

Define (for $j = 1, 2, \ldots, r$)

$$\delta_j = a_{[k_j]}^{(j)} - a_{[k_j - 1]}^{(j)}.$$

Then set $\underset{\sim}{\delta}^* = (\delta_1^*, \delta_2^*, \ldots, \delta_r^*)$ and P^* $(0 < \delta_1^*, \delta_2^*, \ldots, \delta_r^* < \infty;$ $1/K < P^* < 1)$ and seek procedures $\mathcal{P}$ which guarantee the probability requirement

$$P(CS \mid \mathcal{P}) \geq P^* \quad \text{whenever} \quad \underset{\sim}{\delta} \geq \underset{\sim}{\delta}^* \quad \text{componentwise.}$$

Three procedures proposed in the literature are the following.

Procedure SP1 (Factorial Procedure). Take N independent observations from each of the K populations. Let $\overline{X}_{i_j}^{(j)}$ denote the (marginal) sample mean of all observations having the jth factor at level i_j ($j = 1, 2, \ldots, r$). For each j, let $\overline{X}_{[1]}^{(j)} \leq \ldots \leq \overline{X}_{[k_j]}^{(j)}$ denote $\overline{X}_1^{(j)}, \ldots, \overline{X}_{k_j}^{(j)}$ in numerical order. Select the level associated with the largest (marginal) sample mean of each factor, i.e. with $\overline{X}_{[k_1]}^{(1)}, \overline{X}_{[k_2]}^{(2)}, \ldots, \overline{X}_{[k_r]}^{(r)}$, and assert that that population (factor combination) is best. (NK observations are used.)

Procedure SP2 (One-at-a-time Procedure). Fix the level of each factor except factor j, take N_j observations at each level of factor j, and compute the sample mean at each such level. Then select the level (of factor j) yielding the largest such

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sample mean. Proceed similarly for $j = 1, \ldots, r$ (keeping "selected" levels fixed for the corresponding factors), taking a set of independent observations at each stage of experimentation. Finally state that the population corresponding to the selected factor levels is best. ($\sum_{j=1}^{r} k_j N_j$ observations are used.)

Procedure SP3 (Interaction Procedure). Take M independent observations from each of the K populations. Select the population yielding the largest sample mean as best. (MK observations are used.)

In order to compare procedures SP1, SP2, SP3 let n_1, n_2, n_3 denote the respective smallest number of observations each procedure needs in order to guarantee the probability requirement. From Bawa (1972) we know

$$n_1 = \min \{NK: \ N \text{ such that } \prod_{j=1}^{r} (1 - e^{-a_j N_j'}) \geq P^*\}$$

and

$$n_2 = \min \{\sum_{j=1}^{r} k_j N_j: \ (N_1, \ldots, N_r) \text{ such that } \prod_{j=1}^{r} (1-e^{-a_j N_j}) \geq P^*\},$$

where (for $j = 1, 2, \ldots, r$)

$$a_j = \frac{(\delta_j^*)^2}{4\sigma^2}, \ N_j' = \frac{NK}{k_j}.$$

Using asymptotic calculus as in Bawa (1972) and Dudewicz (1969) it follows that

$$n_1 \sim -4\sigma^2 \log(1-P^*) \max_{1 \leq j \leq r} \{\frac{k_j}{(\delta_j^*)^2}\} + o(1-P^*),$$

$$n_2 \sim -4\sigma^2 \log(1-P^*) \sum_{j=1}^{r} \frac{k_j}{(\delta_j^*)^2} + B$$

where $a \sim b$ means the limit of a/b is 1 as $P^* \to 1$ and B does not depend on P^*. Also (Dudewicz (1969))

$$n_3 \sim -4\sigma^2 \log(1-P^*) \frac{K}{(\delta^*)^2},$$

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where $P(CS|SP3) \geq P*$ is guaranteed for $\delta \equiv \mu_{[K]} - \mu_{[K-1]} \geq \delta*$. Therefore

$$ARE(SP1, SP2) = \lim_{P* \to 1} \frac{n_1}{n_2} = \frac{\max_{1 \leq j \leq r} \left(\frac{k_j}{(\delta_j*)^2} \right)}{\sum_{j=1}^{r} \frac{k_j}{(\delta_j*)^2}} \, ,$$

$$ARE(SP1, SP3) = \lim_{P* \to 1} \frac{n_1}{n_3} = \frac{(\delta*)^2}{K} \max_{1 \leq j \leq r} \left(\frac{k_j}{(\delta_j*)^2} \right),$$

$$ARE(SP2, SP3) = \lim_{P* \to 1} \frac{n_2}{n_3} = \frac{(\delta*)^2}{K} \sum_{j=1}^{r} \frac{k_j}{(\delta_j*)^2} \, .$$

Note that

$$\delta = \mu_{[K]} - \mu_{[K-1]} = \left\{ \mu + \sum_{j=1}^{r} \alpha_{[k_j]}^{(j)} \right\} - \max_{1 \leq i \leq r} \left\{ \mu + \sum_{j=1}^{r} \alpha_{[k_j]}^{(j)} - \delta_i \right\}$$

$$= \min \{\delta_1, \ldots, \delta_r\},$$

hence $\underset{\sim}{\delta} \geq \underset{\sim}{\delta}*$ and $\delta* = \min(\delta_1*, \ldots, \delta_r*)$ furnish comparable probability requirements for SP1, SP2, and SP3. Letting c be that j ($1 \leq j \leq r$) where $k_j/(\delta_j*)^2$ is maximized,

$$ARE(SP1, SP3) = \frac{(\delta*)^2}{(\delta_c*)^2} \frac{k_c}{K} \leq 1$$

(since $\delta* \leq \delta_c*$ and $k_c \leq K$), and

$$ARE(SP2, SP3) \leq \frac{(\delta*)^2}{(\delta_c*)^2} \frac{r \, k_c}{K} \leq 1 \, .$$

Hence SP1 and SP2 are each more efficient than SP3. In the "symmetric" case $k_1 = k_2 = \ldots = k_r = k$ and $\delta_1* = \delta_2* = \ldots = \delta_r* = \delta*$

$$ARE(SP1, SP2) = \frac{1}{r}, \ ARE(SP1, SP3) = \frac{1}{k^{r-1}} \, ,$$

$$ARE(SP2, SP3) = \frac{r}{k^{r-1}} \, .$$

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Thus, when there is no interaction, SP1 is best and
SP3 is worst. However, we see (Section 4) that
SP3 is fully resistant to interaction, whereas SP1 and SP2
are not.

4. RANKING AND SELECTION IN FACTORIAL EXPERIMENTS: INTERACTION

We will now study factorial experiments with interactions.
So, we suppose an r-factor $(r \geq 2)$ factorial experiment
with k_j levels of factor j $(j = 1, \ldots, r)$ and assume our
observations are normally distributed with known common
variance $\sigma^2 > 0$ and that if $X_{i_1 i_2 \ldots i_r}$ is an observation taken
at {level i_1 of factor 1, level i_2 of factor 2, $\ldots$, level i_r
of factor r} then

$$E(X_{i_1 i_2 \ldots i_r}) = \mu + \sum_{j_1=1}^{r} \alpha_{i_{j_1}}^{(j_1)} + \sum_{j_2=1}^{r} \sum_{j_1=1}^{r} \alpha_{i_{j_1} i_{j_2}}^{(j_1 j_2)} + \ldots$$

$$+ \sum_{j_r=1}^{r} \sum_{j_r=2}^{r} \ldots \sum_{j_1=1}^{r} \alpha_{i_{j_1} i_{j_2} \ldots i_{j_{r-1}}}^{(j_1 j_2 \ldots j_{r-1})} + \alpha_{i_1 i_2 \ldots i_r}^{(12 \ldots r)}$$

$(i_j = 1, 2, \ldots, k_j \quad$ for $j = 1, 2, \ldots, r)$ where μ and the
α's are unknown parameters with

$$\sum_{i_{j_1}=1}^{k_{j_1}} \alpha_{i_{j_1}}^{(j_1)} = 0 \qquad \text{for } j_1 = 1, 2, \ldots, r ;$$

$$\sum_{i_{j_1}=1}^{k_{j_1}} \alpha_{i_{j_1} i_{j_2}}^{(j_1 j_2)} = \sum_{i_{j_2}=1}^{k_{j_2}} \alpha_{i_{j_1} i_{j_2}}^{(j_1 j_2)} = 0$$

(for $j_1 (=1, 2, \ldots, r) < j_2 (=1, 2, \ldots, r)$);

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$$\sum_{i_{j_1}=1}^{k_{j_1}} \alpha_{i_{j_1} i_{j_2} i_{j_3}}^{(j_1 j_2 j_3)} = \sum_{i_{j_2}=1}^{k_{j_2}} \alpha_{i_{j_1} i_{j_2} i_{j_3}}^{(j_1 j_2 j_3)} = \sum_{i_{j_3}=1}^{k_{j_3}} \alpha_{i_{j_1} i_{j_2} i_{j_3}}^{(j_1 j_2 j_3)} = 0$$

(for $j_1(=1, 2, \ldots, r) < j_2(=1, 2, \ldots, r) < j_3(=1, 2, \ldots, r))$;
and so on; and lastly

$$\sum_{i_1=1}^{k_1} \alpha_{i_1 i_2 \ldots i_r}^{(1\ 2\ldots r)} = \sum_{i_2=1}^{k_2} \alpha_{i_1 i_2 \ldots i_r}^{(1\ 2\ldots r)} = \ldots = \sum_{i_r=1}^{k_r} \alpha_{i_1 i_2 \ldots i_r}^{(1\ 2\ldots r)} = 0$$

($i_j = 1, 2, \ldots, k_j$ for $j = 1, 2, \ldots, r$). For each fixed
j ($j = 1, 2, \ldots, r$), let

$$\alpha_{[1]}^{(j)} \le \alpha_{[2]}^{(j)} \le \cdots \le \alpha_{[k_j]}^{(j)}$$

denote the $\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_{k_j}^{(j)}$ in numerical order.

This r-factor experiment may be conceived of as an r-dimensional cuboid. Each factor-level combination
$(i_1, i_2, \ldots, i_r)$ is called a population, and a "best"
population (among all $K = k_1 k_2 \ldots k_r$ populations) is any one
with $E(X_{i_1 i_2 \ldots i_r})$ maximized over all r-tuples $(i_1, i_2, \ldots, i_r)$ (called cells).

Once observations have been taken, let $\overline{X}_{i_1 i_2 \ldots i_r}$
denote the sample mean of all observations in cell
$(i_1, i_2, \ldots, i_r)$, and define the (marginal) sample means

$$\overline{X}_{i_1 \cdots \cdots} = \frac{\sum_{i_2=1}^{k_2} \sum_{i_3=1}^{k_3} \cdots \sum_{i_r=1}^{k_r} \overline{X}_{i_1 i_2 \ldots i_r}}{k_2 k_3 \ldots k_r} \quad (i_1 = 1, 2, \ldots, k_1),$$

$$\bar{X}.i_2..... = \frac{\sum\limits_{i_1=1}^{k_1} \sum\limits_{i_3=1}^{k_3} \cdots \sum\limits_{i_r=1}^{k_r} \bar{X}_{i_1 i_2 \ldots i_r}}{k_1 k_3 \ldots k_r} \quad (i_2 = 1, 2, \ldots, k_2),$$

.
.
.

$$\bar{X}.......i_r = \frac{\sum\limits_{i_1=1}^{k_1} \sum\limits_{i_2=1}^{k_2} \cdots \sum\limits_{i_{r-1}=1}^{k_{r-1}} \bar{X}_{i_1 i_2 \ldots i_r}}{k_1 k_2 \ldots k_{r-1}} \quad (i_r = 1, 2, \ldots, k_r).$$

We now wish to prove (as indicated briefly in Section 3) that
SP1 and SP2 are vitiated by interaction, while SP3's validity
remains unaltered.

<u>Theorem 1</u>. Let n_1 be the smallest sample size for which
procedure SP1 guarantees $P(CS|SP1) \geq P^*$ whenever $\delta \geq \delta^*$ in
a model with zero interactions. Then in a general (non-
zero interaction) model,

$$\inf_{\Omega(\delta^*)} P(CS|SP1) \leq \frac{1}{K}$$

where $\Omega(\delta^*) = \{(\mu_1, \mu_2, \ldots, \mu_K): \mu_{[K]} - \mu_{[K-1]} \geq \delta^*,$
$\delta^* = \min\{\delta_1^*, \ldots, \delta_r^*\}\}$.

Before proceeding to prove Theorem 1, we will show
that (in the presence of interaction) not only may the cell
selected by SP1 not be the best cell, but it may be the
case (with high probability) that, for each of the r
factors, no column with the respective largest (marginal)
sample mean contains the best cell. The proof will utilize
an addition of interaction to achieve such a situation.

To verify the claims of the above paragraph, consider
a model with no interactions. By labeling and relabeling
the cells, we may arrange our experiment so that

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$$E(X_{i_1 i_2 \ldots i_r}) = \mu + \alpha^{(1)}_{[i_1]} + \alpha^{(2)}_{[i_2]} + \ldots + \alpha^{(r)}_{[i_r]}$$

$$(i_1 = 1, 2, \ldots, k_1; \quad i_2 = 1, 2, \ldots, k_2; \ldots; \quad i_r = 1, 2, \ldots, k_r),$$

still with $\sum_{i_{j_1}=1}^{k_{j_1}} \alpha^{(j_1)}_{i_{j_1}} = 0$ for $j_1 = 1, 2, \ldots, r$. (Thus, cell $(1, 1, \ldots, 1)$ is now the worst cell, while cell $(k_1, k_2, \ldots, k_r)$ is the best.)

Now, to achieve the desired situation, we assume without loss of generality that $k_j \geq k_1 \geq 2$ $(j = 2, 3, \ldots, r)$ and that all interactions are zero except for highest-order interactions. For the case $k_1 = 2$, see Remark 2 following the proof of Theorem 1. In the case $k_1 > 2$, we add highest-order interactions to each $E(X_{i_1 i_2 \ldots i_r})$ as multiples of a number "a" (yet to be chosen) according to a scheme such that cell $(1, 1, \ldots, 1)$ becomes the best cell for large positive "a". To motivate this scheme for r-factor experiments, we first illustrate it for two-factor experiments, next show how to extend it to three-factor experiments, and finally explain the general case $(r > 3)$.

For a two-factor experiment $(r = 2)$, add multiples of a (yet to be chosen) to each $E(X_{i_1 i_2})$ as specified in Figure 1 to form model $E(X_{i_1 i_2}(a))$. Note that there is a simple pattern over most of the table of Figure 1 (except for row 1 and column 1) and that (except for cell (1,1)) row 1 and column 1 entries are chosen so the appropriate row or column sum is zero. Finally, the cell (1,1) entry is chosen so that the row 1 and column 1 sums are each zero also. The added multiples of "a" are the interactions

$\alpha^{(12)}_{i_1 i_2}(a)$, and satisfy $\sum_{i_1=1}^{k_1} \alpha^{(12)}_{i_1 i_2}(a) = \sum_{i_2=1}^{k_2} \alpha^{(12)}_{i_1 i_2}(a) = 0$

Figure 1. Multiples of "a" to be added to each $E(X_{ij})$ to form model $E(X_{ij}(a))$.

	1	2	$\cdots$	k_2-k_1+1	k_2-k_1+2	k_2-k_1+3	$\cdots$	k_2-2	k_2-1	k_2
1	$\dfrac{(k_1-1)k_1(k_1+1)}{6}$	0	$\cdots$	-1	-1	-3	$\cdots$	$-\dfrac{(k_1-3)(k_1-2)}{2}$	$-\dfrac{(k_1-2)(k_1-1)}{2}$	$\dfrac{(k_1-1)k_1}{2}$
2	$-\dfrac{(k_1-1)k_1}{2}$	0	$\cdots$	1	1	2	$\cdots$	k_1-3	k_1-2	k_1-1
3	$-\dfrac{(k_1-2)(k_1-1)}{2}$	0	$\cdots$	0	0	1	$\cdots$	k_1-4	k_1-3	k_1-2
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
k_1-2	-6	0	$\cdots$	0	0	0	$\cdots$	1	2	3
k_1-1	-3	0	$\cdots$	0	0	0	$\cdots$	0	1	2
k_1	-1	0	$\cdots$	0	0	0	$\cdots$	0	0	1

k_2-k_1+1 ... columns; k_1-1 columns

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for all a. (Note that cell (1,1) is best for large positive
a.)

Now "a" can be chosen so that, letting $f_{i_1 i_2}(k_1, k_2)$ be
the Figure 1 entry in row i_1 and column i_2, we
simultaneously satisfy

$$\mu + \alpha^{(1)}_{[1]} + \alpha^{(2)}_{[1]} + \alpha^{(12)}_{11}(a) \geq \mu + \alpha^{(1)}_{[i_1]} + \alpha^{(2)}_{[i_2]} + \alpha^{(12)}_{i_1 i_2}(a) + \delta^* (\forall (i_1, i_2) \neq (1,1)),$$

i.e.

$$\mu + \alpha^{(1)}_{[1]} + \alpha^{(2)}_{[1]} + f_{11}(k_1, k_2) a \geq \mu + \alpha^{(1)}_{[i_1]} + \alpha^{(2)}_{[i_2]} + f_{i_1 i_2}(k_1, k_2) a + \delta^* (\forall (i_1, i_2) \neq (1,1)),$$

i.e.

$$a \geq \frac{\{\alpha^{(1)}_{[i_1]} + \alpha^{(2)}_{[i_2]}\} - \{\alpha^{(1)}_{[1]} + \alpha^{(2)}_{[1]}\} + \delta^*}{f_{11}(k_1, k_2) - f_{i_1 i_2}(k_1, k_2)} \quad (\forall (i_1, i_2) \neq (1,1))$$

(where it is evident from Figure 1 that the denominator is > 0).
For this purpose it suffices to take

$$a = 6 \frac{\{\alpha^{(1)}_{[k_1]} + \alpha^{(2)}_{[k_2]}\} - \{\alpha^{(1)}_{[1]} + \alpha^{(2)}_{[1]}\} + \delta^*}{(k_1 - 1)(k_1 - 2)(k_1 + 3)}.$$

For a <u>three-factor experiment</u> (r = 3),

$$E(X_{i_1 i_2 i_3}) = \mu + \alpha^{(1)}_{i_1} + \alpha^{(2)}_{i_2} + \alpha^{(3)}_{i_3} + \alpha^{(12)}_{i_1 i_2} + \alpha^{(13)}_{i_1 i_3} + \alpha^{(23)}_{i_2 i_3} + \alpha^{(123)}_{i_1 i_2 i_3}$$

($i_1 = 1, 2, \ldots, k_1;\ i_2 = 1, 2, \ldots, k_2;\ i_3 = 1, 2, \ldots, k_3$),

where μ and α's are unknown parameters with

$$\sum_{i_1 = 1}^{k_1} \alpha^{(1)}_{i_1} = \sum_{i_2 = 1}^{k_2} \alpha^{(2)}_{i_2} = \sum_{i_3 = 1}^{k_3} \alpha^{(3)}_{i_3} = 0,$$

$$\sum_{i_1 = 1}^{k_1} \alpha^{(12)}_{i_1 i_2} = \sum_{i_2 = 1}^{k_2} \alpha^{(12)}_{i_1 i_2} = \sum_{i_1 = 1}^{k_1} \alpha^{(13)}_{i_1 i_3} = \sum_{i_3 = 1}^{k_3} \alpha^{(13)}_{i_1 i_3} = \sum_{i_2 = 1}^{k_2} \alpha^{(23)}_{i_2 i_3} = \sum_{i_3 = 1}^{k_3} \alpha^{(23)}_{i_2 i_3} = 0,$$

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$$\sum_{i_1=1}^{k_1} \alpha_{i_1 i_2 i_3}^{(123)} = \sum_{i_2=1}^{k_2} \alpha_{i_1 i_2 i_3}^{(123)} = \sum_{i_3=1}^{k_3} \alpha_{i_1 i_2 i_3}^{(123)} = 0.$$

Figure 2.   Three-factor experiment in the form of a cuboid.
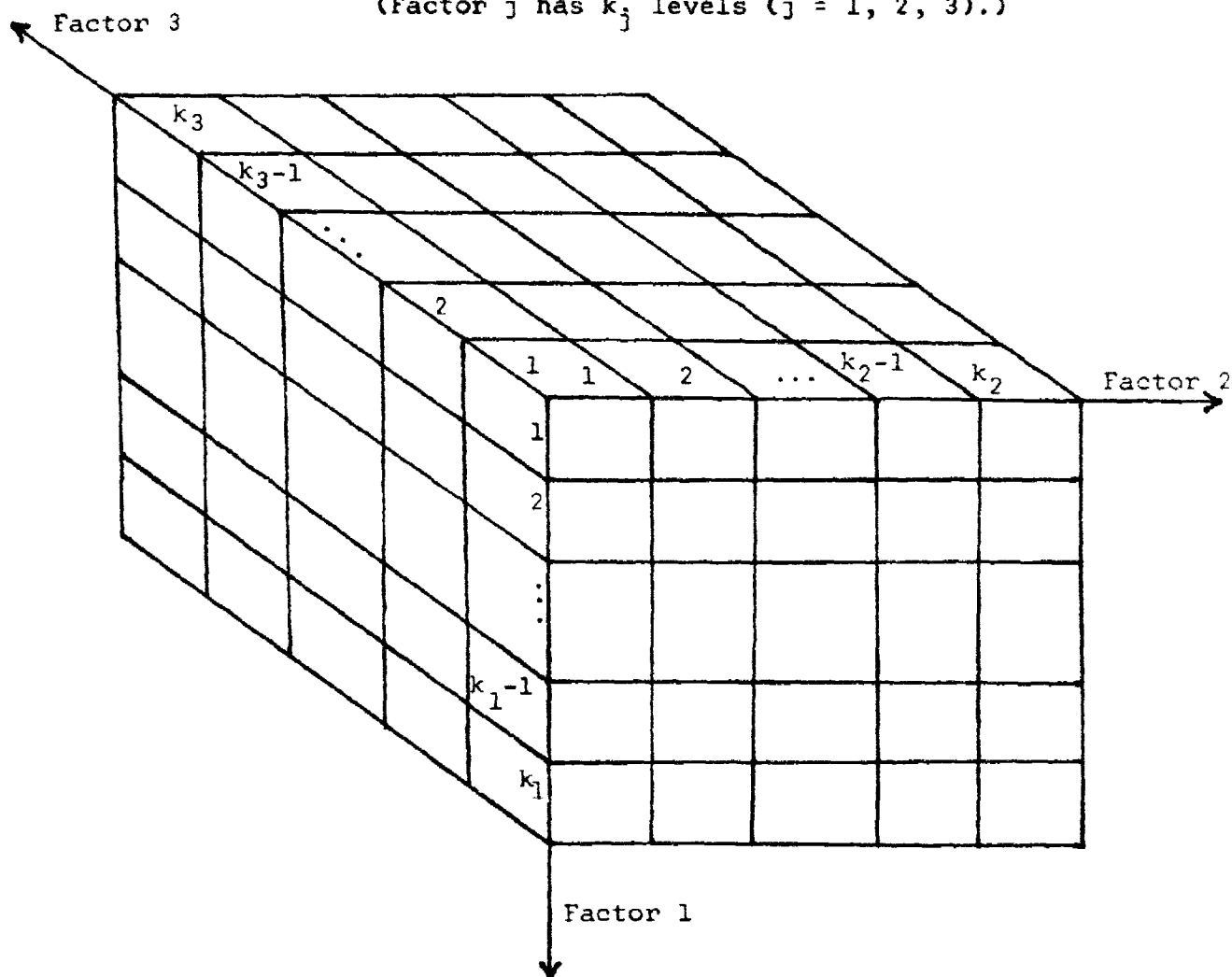(Factor j has k_j levels (j = 1, 2, 3).)

Figure 2 shows the three-factor experiment as a cuboid. We now add second-order interactions to the cells as multiples of a (yet to be chosen), as follows. In the last k_1-1 of the k_3 levels of factor 3 (i.e. levels k_3-k_1+2, ..., k_3 of factor 3) add $-f_{i_1 i_2}(k_1, k_2)a$ in cell (i_1, i_2, i_3),

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where $f_{i_1 i_2}(k_1, k_2)$ is as specified in Figure 1. The other levels of factor 3 have zero second-order interactions in each cell, except for the first level, in which we add $f_{i_1 i_2}(k_1, k_2)a(k_1-1)$ in cells $(i_1, i_2, 1)$, where again $f_{i_1 i_2}(k_1, k_2)$ is as specified in Figure 1. This makes the column sums (corresponding to factor 3) zero, and the interactions all satisfy the model restrictions. (Note that cell $(1, 1, 1)$ is best for large positive a.)

Let $f_{i_1 i_2 i_3}(k_1, k_2, k_3)$ denote the number of multiples of a added to cell (i_1, i_2, i_3) $(i_1 = 1, \ldots, k_1;$ $i_2 = 1, \ldots, k_2;$ $i_3 = 1, \ldots, k_3)$. Then a can be chosen so that we simultaneously satisfy

$$\mu + \alpha_{[1]}^{(1)} + \alpha_{[1]}^{(2)} + \alpha_{[1]}^{(3)} + f_{111}(k_1,k_2,k_3)a \ge \mu + \alpha_{[i_1]}^{(1)} + \alpha_{[i_2]}^{(2)} + \alpha_{[i_3]}^{(3)}$$

$$+ f_{i_1 i_2 i_3}(k_1,k_2,k_3)a + \delta^* (\forall(i_1,i_2,i_3) \neq (1, 1, 1)),$$

i.e.

$$a \ge \frac{\{\alpha_{[i_1]}^{(1)} + \alpha_{[i_2]}^{(2)} + \alpha_{[i_3]}^{(3)}\} - \{\alpha_{[1]}^{(1)} + \alpha_{[1]}^{(2)} + \alpha_{[1]}^{(3)}\} + \delta^*}{f_{111}(k_1,k_2,k_3) - f_{i_1 i_2 i_3}(k_1,k_2,k_3)} \quad (\forall(i_1,i_2,i_3) \neq (1,1,1))$$

(where the denominator is > 0).
For this purpose it suffices to take

$$a = 6 \frac{\{\alpha_{[k_1]}^{(1)} + \alpha_{[k_2]}^{(2)} + \alpha_{[k_3]}^{(3)}\} - \{\alpha_{[1]}^{(1)} + \alpha_{[1]}^{(2)} + \alpha_{[1]}^{(3)}\} + \delta^*}{(k_1-2)(k_1-1)k_1(k_1+2)}.$$

The general case of an r-factor experiment $(r \ge 2)$ is handled by induction on r, using the method shown to proceed from $r = 2$ to $r = 3$ above. Namely, suppose we have an interaction scheme for an $(r-1)$-factor experiment where all the interactions are zero except the highest-order ones. In order to extend it to an r-factor experiment, suppose that the rth factor has k_r levels. In the last

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(k_1-1) levels of factor r (that is, from k_r-k_1+2 through k_r, both inclusive), we add interactions of highest-order as in the (r-1)-factor experiment scheme, but with sign of each entry reversed. The rest of the levels (except for the first one) receive zero highest-order interaction in each cell. Then to make appropriate interaction sums zero, we add highest-order interactions in the first level in such a way that each entry is (k_1-1) times the corresponding entry of the (r-1)-factor experiment scheme. As before we may find a large positive number a such that cell (1, 1, ..., 1) is best.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let n_1 for SP1 satisfy $P(CS|SP1) \geq P^*$ whenever $\delta \geq \delta^*$ in a model with zero interactions (which, of course, implies a sample size N per cell, with $n_1 = NK$). Now $\overline{X}_{1......}$, $\overline{X}_{2......}$, ..., $\overline{X}_{k_1......}$ are independent normal random variables with (possibly different) means and the same variance $k_1\sigma^2/(NK)$, hence

$$\overline{X}_{1......}^*, \quad \overline{X}_{2......}^*, \quad ..., \quad \overline{X}_{k_1......}^*$$

are independent N(0,1) random variables when we define

$$\overline{X}_{i_1......}^* = \frac{\overline{X}_{i_1......} - E(\overline{X}_{i_1......})}{\sqrt{\dfrac{\sigma^2 k_1}{NK}}} , \quad (i_1 = 1, 2, ..., k_1).$$

Now consider model $E(X_{i_1 i_2 ... i_r}(a))$ with "a" chosen so large positive that cell (1, 1, ..., 1) is best, according to the interaction scheme previously given. Then, in this model with interaction,

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$$P(CS|SP1) = P[\overline{X}_{1\ldots\ldots} = \max(\overline{X}_{1\ldots\ldots}, \overline{X}_{2\ldots\ldots}, \ldots, \overline{X}_{k_1\ldots\ldots});$$

$$\overline{X}_{.1\ldots\ldots} = \max(\overline{X}_{.1\ldots\ldots}, \overline{X}_{.2\ldots\ldots}, \ldots, \overline{X}_{.k_2\ldots\ldots});$$

$$\ldots$$

$$\overline{X}_{\ldots\ldots1} = \max(\overline{X}_{\ldots\ldots1}, \overline{X}_{\ldots\ldots2}, \ldots, \overline{X}_{\ldots\ldots k_r})]$$

$$= P[Y_{11} > 0, Y_{12} > 0, \ldots, Y_{1,k_1-1} > 0;$$

$$Y_{21} > 0, Y_{22} > 0, \ldots, Y_{2,k_2-1} > 0;$$

$$\ldots$$

$$Y_{r1} > 0, Y_{r2} > 0, \ldots, Y_{r,k_r-1} > 0]$$

where

$$Y_{1i_1} = \overline{X}_{1\ldots\ldots} - \overline{X}_{i_1+1,\ldots\ldots} \quad (i_1 = 1, 2, \ldots, k_1-1),$$

$$Y_{2i_2} = \overline{X}_{.1\ldots\ldots} - \overline{X}_{.,i_2+1,\ldots\ldots} \quad (i_2 = 1, 2, \ldots, k_2-1),$$

$$\ldots$$

$$Y_{ri_r} = \overline{X}_{\ldots\ldots1} - \overline{X}_{\ldots\ldots,i_r+1} \quad (i_r = 1, 2, \ldots, k_r-1).$$

Now $(Y_{11}, \ldots, Y_{1,k_1-1}; Y_{21}, \ldots, Y_{2,k_2-1}; \ldots; Y_{r1}, \ldots, Y_{r,k_r-1})$

has a $(k_1 + k_2 + \ldots + k_r - r)$-variate normal distribution with

$$E(Y_{1i_1}) = \alpha^{(1)}_{[i_1]} - \alpha^{(1)}_{[i_1+1]}, \quad Var(Y_{1i_1}) = \frac{2\sigma^2 k_1}{NK} \quad (i_1 = 1, 2, \ldots, k_1-1),$$

$$E(Y_{2i_2}) = \alpha^{(2)}_{[i_2]} - \alpha^{(2)}_{[i_2+1]}, \quad Var(Y_{2i_2}) = \frac{2\sigma^2 k_2}{NK} \quad (i_2 = 1, 2, \ldots, k_2-1),$$

$$\ldots$$

$$E(Y_{ri_r}) = \alpha^{(r)}_{[i_r]} - \alpha^{(r)}_{[i_r+1]}, \quad Var(Y_{ri_r}) = \frac{2\sigma^2 k_r}{NK} \quad (i_r = 1, 2, \ldots, k_r-1),$$

$$Cov(Y_{1i_1}, Y_{1i_1'}) = \frac{\sigma^2 k_1}{NK} \quad (i_1 \neq i_1'),$$

$$Cov(Y_{2i_2}, Y_{2i_2'}) = \frac{\sigma^2 k_2}{NK} \quad (i_2 \neq i_2'),$$

...

$$\text{Cov}(Y_{ri_r}, Y_{ri_r'}) = \frac{\sigma^2 k_r}{NK} \ (i_r \neq i_r'),$$

$$\text{Cov}(Y_{ji_j}, Y_{\ell i_\ell}) = 0 \quad (j \neq \ell; \ j = 1, \ldots, r; \ \ell = 1, \ldots, r)$$

$$(i_1 = 1, 2, \ldots, k_1-1; \ i_2 = 1, 2, \ldots, k_2-1; \ldots; \ i_r = 1, 2, \ldots, k_r-1).$$

Thus $(Y_{11}, \ldots, Y_{1,k_1-1}), (Y_{21}, \ldots, Y_{2,k_2-1}), \ldots, (Y_{r1}, \ldots, Y_{r,k_r-1})$ are independent, so

$$P(CS|SP1) = \prod_{j=1}^{r} P[Y_{j1} > 0, Y_{j2} > 0, \ldots, Y_{j,k_j-1} > 0].$$

Considering the first factor,

$$P[Y_{11} > 0, Y_{12} > 0, \ldots, Y_{1,k_1-1} > 0]$$

$$= P[\bar{X}_{1\ldots} = \max(\bar{X}_{1\ldots}, \bar{X}_{2\ldots}, \ldots, \bar{X}_{k_1\ldots})]$$

$$\leq P[\bar{X}^*_{1\ldots} = \max(\bar{X}^*_{1\ldots}, \bar{X}^*_{2\ldots}, \ldots, \bar{X}^*_{k_1\ldots})]$$

$$= \frac{1}{k_1} .$$

Proceeding similarly, we find that

$$P[Y_{j1} > 0, Y_{j2} > 0, \ldots, Y_{j,k_j-1} > 0] \leq \frac{1}{k_j} \ (j = 2, \ldots, r),$$

hence

$$P(CS|SP1) \leq 1/K$$

as was to be proven.

Remark 1. Note that for all schemes of adding interactions and for all $k_j \geq 2$ $(j = 1, 2, \ldots, r)$, procedure SP1 still achieves probability at least P^* for selecting the cell associated with $\mu + \sum_{j=1}^{r} \alpha^{(j)}_{[k_j]}$ (which may not be the "best"

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cell after adding interactions). This is true because
the sum of interactions in each row and in each column
is zero and so the (marginal) sample means do not change.

Remark 2. In the case $k_1 = 2$, we add interactions in
such a way that cell $(k_1, k_2, \ldots, k_r)$ does not remain
best. From Remark 1, it then follows that

$$P(CS|SP1) \leq 1-P^* .$$

Theorem 2. Let n_2 be the smallest sample size for which
procedure SP2 guarantees $P(CS|SP2) \geq P^*$ whenever $\underset{\sim}{\delta} \geq \underset{\sim}{\delta}^*$
in a model with zero interactions. Then in a general
(non-zero interaction) model,

$$\underset{\Omega(\delta^*)}{\inf} P(CS|SP2) \leq \frac{1}{\underset{1\leq j\leq r}{\min} k_j}$$

where $\Omega(\delta^*) = \{(\mu_1,\mu_2,\ldots,\mu_K):\mu_{[K]}-\mu_{[K-1]}\geq\delta^*,\delta^*=\min\{\delta_1^*,\ldots,\delta_r^*\}\}$.

Proof of Theorem 2. Let n_2 for SP2 satisfy $P(CS|SP2) \geq P^*$
whenever $\underset{\sim}{\delta} \geq \underset{\sim}{\delta}^*$ in a model with zero interactions. Let factors
be ordered as in the discussion following the statement of
Theorem 1, and consider model $E(X_{i_1 i_2 \ldots i_r}(a))$ with a set so
that cell $(1, 1, \ldots, 1)$ is best. Suppose, without loss of
generality, that the experimenter decides to first fix levels
of each factor except factor j (and hence experiments across
the levels of factor j). Define events

F_{ji_j} = {Experimenter starts with level i_j of factor j},

E_{ji_j} = {Experimenter selects level i_j of factor j},

$(i_j = 1, \ldots, k_j; j = 1, \ldots, r)$ so that event $E_{11}E_{21}\ldots E_{r1}$
corresponds to a correct selection. Now, for any $\epsilon > 0$, a may
be taken sufficiently large that

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$$P(E_{11} E_{21} \cdots E_{r1} | F_{j i_j}) \leq \epsilon \qquad (i_j \neq 1).$$

Now, letting $F_j = F_{j1} \cup \ldots \cup F_{jk_j}$ (the event that the experimenter starts with factor j) and taking $P(F_{j1}|F_j) = \ldots = P(F_{jk_j}|F_j) = 1/k_j$ as a reasonable model in light of no prior knowledge (on the part of the experimenter) as to which levels are better than others, we find

$$P(CS|SP2, F_j) = \sum_{i_j=1}^{k_j} P(E_{11} E_{21} \cdots E_{r1} F_{j i_j}|F_j)$$

$$= \sum_{i_j=1}^{k_j} P(F_{j i_j}|F_j) P(E_{11} E_{21} \cdots E_{r1}|F_{j i_j})$$

$$= P(F_{j1}|F_j) P(E_{11}E_{21}\cdots E_{r1}|F_{j1}) + \sum_{i_j=2}^{k_j} P(F_{j i_j}|F_j) P(E_{11}E_{21}\cdots E_{r1}|F_{j i_j})$$

$$\leq \frac{1}{k_j} + \sum_{i_j=2}^{k_j} \frac{\epsilon}{k_j} \leq \frac{1}{k_j} + \epsilon,$$

and the theorem follows.

<u>Theorem 3.</u> Let n_3 be the smallest sample size for which procedure SP3 guarantees $P(CS|SP3) \geq P^*$ whenever $\underset{\sim}{\delta} \geq \underset{\sim}{\delta^*}$ in a model with zero interactions. Then in a general (non-zero interaction) model,

$$P(CS|SP3) \geq P^* \text{ whenever } \mu_{[K]} - \mu_{[K-1]} \geq \delta^* = \min\{\delta_1^*, \ldots, \delta_r^*\}.$$

Proof of Theorem 3.   In a model with zero interactions, suppose that cell $(m_1, m_2, \ldots, m_r)$ is best.   Then, if M observations per cell have been taken, whenever $\underset{\sim}{\delta} \geq \underset{\sim}{\delta^*}$ componentwise we have

$$P(CS|SP3) = P[\overline{X}_{m_1 m_2 \ldots m_r} = \underset{i_1, i_2, \ldots, i_r}{\max} \{\overline{X}_{i_1 i_2 \ldots i_r}\},$$

$$i_j = 1, 2, \ldots, k_j \ (j = 1, 2, \ldots, r)]$$

$$= P[\overline{X}^*_{i_1 i_2 \ldots i_r} \leq \overline{X}^*_{m_1 m_2 \ldots m_r} + \frac{E(\overline{X}_{m_1 m_2 \ldots m_r}) - E(\overline{X}_{i_1 i_2 \ldots i_r})}{\sqrt{\frac{\sigma^2}{M}}},$$

$$i_j = 1, 2, \ldots, k_j \ (j = 1, 2, \ldots, r)]$$

$$\geq P[\overline{X}^*_{i_1 i_2 \ldots i_r} \leq \overline{X}^*_{m_1 m_2 \ldots m_r} + \frac{\min\{\delta^*_1, \ldots, \delta^*_r\}}{\sqrt{\frac{\sigma^2}{M}}},$$

$$i_j = 1, 2, \ldots, k_j \ (j = 1, 2, \ldots, r)]$$

and

$$\underset{\underset{\sim}{\delta} \geq \underset{\sim}{\delta^*}}{\inf} P(CS|SP3) = \int_{-\infty}^{\infty} \phi^{K-1}(x + \frac{\sqrt{M} \min\{\delta^*_1, \ldots, \delta^*_r\}}{\sigma}) \phi(x) \, dx.$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal distribution and density functions, respectively.   For this inf to be $\geq P^*$ we need

$$M \geq \frac{h_K^2(P^*)\sigma^2}{(\min\{\delta^*_1, \ldots, \delta^*_r\})^2}$$

where $h_K(P^*)$ is the solution h of the equation

$$\int_{-\infty}^{\infty} \phi^{K-1}(x+h)\phi(x)dx = P^* .$$

Now in a general (non-zero interaction) model suppose that cell $(c_1, c_2, \ldots, c_r)$ is best, M as above is used, and $\mu_{[K]} - \mu_{[K-1]} \geq \delta^* = \min\{\delta^*_1, \ldots, \delta^*_r\}$.   Then

$$P(CS|SP3) = P[\overline{X}^*_{i_1 i_2 \ldots i_r} \leq \overline{X}^*_{c_1 c_2 \ldots c_r} + \frac{E(\overline{X}_{c_1 c_2 \ldots c_r}) - E(\overline{X}_{i_1 i_2 \ldots i_r})}{\sqrt{\frac{\sigma^2}{M}}},$$

$$i_j = 1, 2, \ldots, k_j \ (j=1, 2, \ldots, r)]$$

$$\geq P[\overline{X}^*_{i_1 i_2 \ldots i_r} \leq \overline{X}^*_{c_1 c_2 \ldots c_r} + \frac{\delta^* \sqrt{M}}{\sigma},$$

$$i_j = 1, 2, \ldots, k_j \ (j = 1, 2, \ldots, r)]$$

$$= \int_{-\infty}^{\infty} \phi^{K-1}(x + \frac{\delta^* \sqrt{M}}{\sigma})\phi(x)dx \geq P^*$$

and the theorem follows.

Thus, while procedure SP3 is fully robust to the presence of interaction, procedures SP1 and SP2 may be fully vitiated by interaction. One may therefore wish to estimate interaction size and choose accordingly between SP1 and SP3 in practice. An SP4 which incorporates this idea (acting as does SP1 when interactions are "negligible" and as does SP3 when interactions are "large") is now being developed.

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